Exercise 13

(a) Find the average rate of change of the area of a circle with respect to its radius r as r changes from

(i) 2 to 3 (ii) 2 to 2.5 (iii) 2 to 2.1

(b) Find the instantaneous rate of change when $r = 2$.

(c) Show that the rate of change of the area of a circle with respect to its radius (at any r) is equal to the circumference of the circle. Try to explain geometrically why this is true by drawing a circle whose radius is increased by an amount Δr. How can you approximate the resulting change in area ΔA if Δr is small?

Solution

Part (a)

The average rate of change of the area with respect to radius is given by the slope of the secant line.

$$\text{(i)} \qquad \frac{\Delta A}{\Delta r} = m = \frac{A(3) - A(2)}{3 - 2} = \frac{\pi(3)^2 - \pi(2)^2}{1} = \pi(9 - 4) = 5\pi$$

$$\text{(ii)} \qquad \frac{\Delta A}{\Delta r} = m = \frac{A(2.5) - A(2)}{2.5 - 2} = \frac{\pi(2.5)^2 - \pi(2)^2}{0.5} = 2\pi(6.25 - 4) = 4.5\pi$$

$$\text{(iii)} \qquad \frac{\Delta A}{\Delta r} = m = \frac{A(2.1) - A(2)}{2.1 - 2} = \frac{\pi(2.1)^2 - \pi(2)^2}{0.1} = 10\pi(4.41 - 4) = 4.1\pi$$

Part (b)

Calculate the derivative of $A(r) = \pi r^2$.

$$A'(r) = 2\pi r$$

Consequently, the instantaneous rate of change when $r = 2$ is

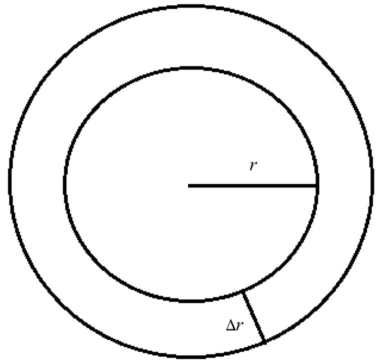
$$A'(2) = 2\pi(2) = 4\pi.$$

Part (c)

Since the circumference C of a circle of radius r is $2\pi r$,

$$A'(r) = C.$$

Suppose there's a circle with radius r, and the radius increases by Δr.



The old area is $A_{\text{old}} = \pi r^2$, and the new area is

$$\begin{aligned} A_{\text{new}} &= \pi(r + \Delta r)^2 \\ &= \pi[r^2 + 2r\Delta r + (\Delta r)^2] \\ &= \pi r^2 + 2\pi r\Delta r + \pi(\Delta r)^2. \end{aligned}$$

Because Δr is assumed to be small, $\pi(\Delta r)^2$ is extremely small compared to $\pi r^2 + 2\pi r\Delta r$ and can be neglected to a good approximation.

$$A_{\text{new}} \approx \pi r^2 + 2\pi r\Delta r$$

Therefore, the approximate change in area is

$$\begin{aligned} \Delta A &= A_{\text{new}} - A_{\text{old}} \\ &\approx (\pi r^2 + 2\pi r\Delta r) - \pi r^2 \\ &\approx 2\pi r\Delta r. \end{aligned}$$